Contract Auctions for Sponsored Search

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Abstract. In sponsored search auctions advertisers typically pay a fixed amount per click that their advertisements receive. In particular, the advertiser and the publisher enter into a contract (e.g., the publisher displays the ad; the advertiser pays the publisher 10 cents per click), and each party's subjective value for such a contract depends on their estimated click-through rates (CTR) for the ad. Starting from this motivating example, we define and analyze a class of contract auctions that generalize the classical second price auction. As an application, we introduce impression-plus-click pricing for sponsored search, in which advertisers pay a fixed amount per impression plus an additional amount if their ad is clicked. Of note, when the advertiser's estimated CTR is higher than the publisher's estimated CTR, both parties find negative click payments advantageous, where the advertiser pays the publisher a premium for the impression but the publisher then pays the advertiser per click.

1 Introduction

In the classical sealed-bid second-price auction, bidders report their value for the auctioned good, and the winner is the bidder with the highest reported value. Incentive compatibility is achieved by charging the winner the least amount for which they would have still won the auction (i.e., the winner pays the second highest bid). In contrast, consider a typical sponsored search auction where, for simplicity, we assume bidders compete for a single available impression: Advertisers report their value-per-click; the winner is the bidder from whom the publisher expects to receive the most revenue; and the winning bidder pays the least amount per click for which they would have still won the auction. While sponsored-search auctions are conceptually similar to traditional second price auctions, there is a key difference: Goods in traditional auctions are exchanged for deterministic payments, and in particular, the value of these payments is identical to the bidder and the auctioneer; in sponsored search auctions, impressions are exchanged for stochastic payments, and the value of such payments to the publisher and the advertiser depends on their respective estimated click-through rates (CTR). For example, if the advertiser's estimated CTR is higher than the publisher's, then the advertiser would expect to pay more than the publisher would expect to receive.

Starting from this motivating example of sponsored search, we define and develop a framework for contract auctions that generalize the second price auction. We consider arbitrary agent valuations over a space of possible contracts; in particular, valuations may diverge for reasons other than mismatched probability estimates. As an application, we introduce impression-plus-click (IPC) sponsored search auctions, in which advertisers pay a fixed amount per impression plus make an additional payment per click. Interestingly, when the advertiser's estimated CTR is higher than the publisher's estimate, both parties prefer negative click payments—or *paid per click* pricing: The advertiser pays the publisher a premium for the impression, and the publisher then pays the advertiser per click.

In the remainder of this introduction we review sponsored search auctions and related work. General contract auctions are developed in Section 2, and a dominant strategy incentive compatible mechanism is proposed. Impression-plus-click sponsored search auctions are introduced in Section 3. In Section 4 we analyze an impression-or-click auction, and consider connections to the hybrid auction model of Goel & Munagala [7]. We conclude in Section 5 by discussing potential offline applications of this work, including applications to insurance, book publication and executive compensation. Some proofs are left to the Appendix.

1.1 Background and Related Work

Sponsored search is the practice of auctioning off ad placement next to web search results; advertisers pay the search engine when their ads are clicked. These ad auctions are responsible for the majority of the revenue of today's leading search engines [13]. Edelman et al. [6] and Varian [18] provide the most basic, standard model for sponsored search auctions and analyze its equilibrium properties (see also Lahaie et al. [14] for a survey of the literature in this area). We do not provide a description of this model here because our contract auction abstracts away from its details in order to cover pricing schemes beyond per-click or per-impression.

Harrenstein et al. [10] recently and independently developed the *qualitative Vickrey auction*, a mechanism similar to the general contract auction presented here. The primary differences between their work and ours concern subtleties in the bidding language, the tie-breaking rules, and the assumptions guaranteeing truthfulness. In this paper we detail our interpretation and results for contract auctions; our main contribution, however, is applying this framework to sponsored search, and in particular introducing and analyzing impression-plus-click pricing.

Truthfulness under the standard model of sponsored search is well understood [1]. In mechanism design more generally, Myerson [16] characterizes payment rules that achieve truthfulness when types are single-dimensional. Holmstrom [11] gives a characterization for type spaces that are smoothly path-connected (see also [15]). In contrast, our truthfulness result for contract auctions does not assume any topology on the type space. Instead it is a consequence of the particular structure of the outcome space (the auctioneer may contract with only one agent) together with a novel consistency condition between the auctioneer and agents' preferences.

Contract auctions generalize the single-item Vickrey auction [19], but are conceptually distinct from the well-known the Vickrey-Clark-Groves (VCG) mechanism [4, 9]. An intuitive interpretation of the VCG mechanism is that it charges each agent the externality that the agent imposes on others; thus, the mechanism only applies when utility is transferable between agents through payments. This is not possible when, for instance, the agents and auctioneer disagree on click-through rates, because there can be no agreement on how to quantify the externality.

Although the basic model of sponsored search given in [6, 18] assumes agreement on click-through rates, there are many reasons why disagreement might arise. Clicks on ads are low probability events and their rate of arrival can be very hard to estimate [17]. Even for ads with a long history of impressions, estimates may diverge because of click fraud, a practice whereby an advertiser clicks on a competitor's ad in order to deplete the latter's budget [8, 12]. If the advertiser and auctioneer differ on which clicks were fraudulent, their click-through rate estimates would then also differ.

The hybrid auction of Goel and Munagala [7] is a notable departure from the basic sponsored search model in that it attempts to reconcile differing publisher and advertiser click-through estimates. In a hybrid auction advertisers place per-click bids as well as per-impression bids, and the auctioneer then chooses one of the two pricing schemes. Goel and Munagala [7] show that, besides being truthful, their hybrid auction has many advantages over simple per-click keyword auctions. The auction allows advertisers to take into account their attitudes towards risk and may generate higher revenue, among other nice properties. The consistency condition given in this work distills the reason behind truthfulness in the hybrid auction, and our contract auction leads to variants and generalizations of the hybrid auction to multiple pricing schemes beyond CPC and CPM (e.g., CPA for any kind of action).

Although our motivation is sponsored search, contract auctions also find potential applications in display advertising. This market includes a variety of different advertisers ranging from brand to direct marketers. Consequently, ad networks typically offer a variety of different pricing schemes including CPM, CPC, and CPA [5]. Amer-Yahia et al. [2] and Boutilier et al. [3] discuss bidder preferences in display advertising.

2 Contract Auctions

We define and develop an incentive compatible mechanism for contract auctions where agents have valuations over an arbitrary space of possible contracts. Suppose there are N agents $A_1, \ldots, A_N$ and finite sets $C_1, \ldots, C_N$ that denote the set of potential contracts each agent could enter into. Agents have valuation functions $v_i : C_i \mapsto \mathbb{R}$ for their respective contracts, and the auctioneer's value for each contract is given by $v_i^{\mathcal{A}} : C_i \mapsto \mathbb{R}$. *Contracts*, in this setting, are nothing more than abstract objects for which each party has a value. The auctioneer is to enter into a single contract, and our goal is to design a framework to facilitate this transaction.

The valuation functions are intended to represent purely subjective utilities, based, for example, on private beliefs or simply taste. In this sense, each agent values contracts in their own "currency," which cannot directly be converted into values for other agents. We require that preferences be *consistent* in the following sense: Among contracts acceptable to a given bidder (i.e., those contracts for which the bidder has non-negative utility), the highest value contract to the auctioneer is one for which the bidder has zero utility. This statement is formalized by Definition 1.

Definition 1. *In the setting above, we say agent v_i and the auctioneer have* consistent *valuations if for each $c_1 \in C_i$ with $v_i(c_1) > 0$, there exists $c_2 \in C_i$ such that $v_i(c_2) \geq 0$ and $v_i^{\mathcal{A}}(c_2) > v_i^{\mathcal{A}}(c_1)$.*

Consistency is equivalent to the following property:

$$\max_{\{c: v_i(c) \geq 0\}} v_i^{\mathcal{A}}(c) > \max_{\{c: v_i(c) > 0\}} v_i^{\mathcal{A}}(c).$$

We note that consistency is a weak restriction on the structure of valuations. In particular, if contracts include a "common currency" component, for which bidders and the auctioneer have an agreed upon value, then valuations are necessarily consistent.

Under this assumption of consistency, Mechanism 1 defines a dominant strategy incentive-compatible mechanism for contract auctions. First, bidders report their valuation function to the auctioneer. In the applications we consider, this entails reporting a small set of parameters which defines the valuation function over the entire contract space. Next, among contracts for which agents have non-negative utility (i.e., "acceptable" or "individually-rational" contracts), the auctioneer identifies the contract for which it has maximum value; the winner of the auction is the bidder who submitted this maximum value acceptable contract. Finally, the auctioneer and the winner enter into the best contract from the winner's perspective for which it would have still won the auction.

Mechanism 1 A General Contract Auction

1: Each agent $A_1, \dots, A_N$ reports a valuation function $\tilde{v}_i$.

2: For $1 \le i \le N$, let $S_i = \{c \in C_i \mid \tilde{v}_i(c) \ge 0\}$ be the set of contracts for which agent A_i claims to have non-negative valuation, and define

$$R_i = \max_{S_i} v_i^{\mathcal{A}}(c) \tag{1}$$

to be the maximum value the auctioneer can achieve from each agent among these acceptable contracts.

3: Fix h so that $R_{h(1)} \ge R_{h(2)} \ge \cdots \ge R_{h(N)}$, and let

$$S = \left\{ c \in C_{h(1)} \,\middle|\, v_{h(1)}^{\mathcal{A}}(c) \ge R_{h(2)} \right\}.$$

With agent $A_{h(1)}$, the auctioneer enters into any contract c^* such that

$$c^* \in \arg\max_{S} \tilde{v}_{h(1)}(c).$$

Theorem 1. *In the setting above, suppose agents have consistent valuations. Then Mechanism 1 is dominant strategy incentive compatible.*

Proof. Fix an agent A_i and consider its strategy. Let $R_{-i} = \max_{j \ne i} R_j$ where R_j is defined as in (1). If A_i were to win the auction, then it would necessarily enter into a contract among those in the set

$$M_i = \left\{ c \in C_i \,\middle|\, v_i^{\mathcal{A}}(c) \ge R_{-i} \right\}.$$

Suppose A_i has strictly positive valuation for some contract $c_1 \in M_i$ (i.e., $\max_{M_i} v_i(c) > 0$). Then by the assumption of consistent valuations, there exists a contract c_2 such that $v_i(c_2) \ge 0$ and

$$v_i^{\mathcal{A}}(c_2) > v_i^{\mathcal{A}}(c_1) \ge R_{-i}.$$

In particular, if A_i truthfully reports its valuation function, then we would have $R_i \ge v_i^{\mathcal{A}}(c_2) > R_{-i}$, and hence A_i would win the auction. Furthermore, in this case the best A_i could do is to enter into a contract in the set $\arg\max_{M_i} v_i(c)$. Again, truthful reporting ensures that this optimal outcome occurs.

Now suppose $\max_{M_i} v_i(c) \leq 0$. In this case A_i has no possibility of positive gain, whether or not it wins the auction. However, by reporting truthfully, if A_i does win the auction the final contract would be selected from the set $S_i = \{c \in C_i \,|\, v_i(c) \geq 0\}$. That is, truthful reporting ensures that A_i achieves (the optimal) zero gain. □

We next show that the consistency condition plays a crucial role in achieving incentive compatibility by exhibiting an example with inconsistent valuations where truth telling is not a dominant strategy. Suppose the auctioneer has one item for sale and there are two agents A_i and A_j. Agent A_i, Irene, values the item at \$4 but only has \$2 to spend. There are three contracts she can enter into, c_1^i, c_2^i, c_3^i, intuitively buying the item for \$1, \$2, and \$3, resulting in utilities of $3, 2$, and -1, the latter being negative since Irene has a limited budget of \$2. Agent A_j, Juliet, values the item at \$2 and has \$2 to spend. She can enter into three similar contracts, c_1^j, c_2^j and c_3^j, resulting in utilities of $1, 0$ and -1 respectively. From the auctioneer's point of view, his utility is the revenue, $v^{\mathcal{A}}(c_x^i) = v^{\mathcal{A}}(c_x^j) = x$ for any $x \in \{1, 2, 3\}$.

If the agents report their valuations truthfully, then $R_i = R_j = 2$ and the auctioneer must break the tie. Unless the tie is broken deterministically in favor of Irene, she has an incentive to lie. Suppose she reports her valuation for c_3^i, as $v_i(c_3^i) = 1$, pretending that she has enough money to afford the item. In that case $R_i = 3$, and $R_j = 2$, so Irene wins the item; but she can select any outcome so long as the auctioneer's utility is at least $R_j = 2$. She chooses c_2^i, which has a positive utility to her, but still makes \$2 for the auctioneer. Essentially, because the utilities of Irene and the auctioneer are not consistent, Irene can bluff to always win the item.

Remark 1. In the above we have assumed the contract spaces C_i are finite. This restriction is imposed only to ensure the *maximum* operation is well-defined in Mechanism 1. We implicitly relax this condition in the following discussion, as it is clear the maxima exist despite having infinite contract spaces.

Mechanism 1 generalizes the usual sealed-bid second-price auction. To see this, take $C_i = \mathbb{R}$, and let the contract $p \in \mathbb{R}$ indicate agent A_i's obligation to purchase the auctioned good at price p. If agent A_i values the good at w_i, then its value over contracts is given by $v_i(p) = w_i - p$, and in particular, its preferences over contracts is parametrized by $w_i \in \mathbb{R}$. The auctioneer has valuation $v_i^{\mathcal{A}}(p) = p$. Now, letting $\tilde{w}_i$ be A_i's reported valuation, we have $R_i = \tilde{w}_i$. Furthermore,

$$S = \left\{ c \in C_{h(1)} \,\middle|\, v_{h(1)}^{A}(c) \geq R_{h(2)} \right\} = \left[\tilde{w}_{h(2)}, \infty\right)$$

and so $\arg\max_S \tilde{v}_{h(1)} = \tilde{w}_{h(2)}$. That is, agent $A_{h(1)}$ enters into the contract $\tilde{w}_{h(2)}$, agreeing to pay the second highest bid for the good.

3 The Impression-Plus-Click Pricing Model

We now consider a specific application of contract auctions for sponsored search: impression-plus-click pricing. For a given impression, define a contract $(r_s, r_f) \in \mathbb{R}^2$ to require the advertiser pay r_s if a click occurs and r_f if no click occurs. This is a complete pricing scheme if the advertiser values only impressions and clicks. We note that so-called "brand advertisers" often have significant utility for simply displaying their ads, regardless of whether or not their ads are clicked. These contracts are equivalently parametrized by $(r_m, r_c) \in \mathbb{R}^2$, where the advertiser pays r_m per impression and an *additional* r_c per click. Using this latter, additive, notation, an impression-plus-click (IPC) contract is represented as point in the CPM-CPC price plane. A priori there are no restrictions on these prices (e.g., one or both coordinates could be negative).

3.1 Contract Preferences

Suppose an advertiser A_i values an impression, regardless of whether it receives a click, at $m_i \geq 0$, values a click at $w_i \geq 0$, and estimates its CTR to be $p_i > 0$. Then, assuming risk neutrality, its value for the IPC contract (r_m, r_c) is

$$v_i(r_m, r_c) = (m_i + p_i w_i) - (r_m + p_i r_c).$$

Observe that the contract preferences of A_i are equivalent to those of an advertiser who values clicks at $w_i + m_i/p_i$ and has no inherent value for impressions. Consequently, without loss of generality, we need only consider the case $m_i = 0$. We thus have the simplified expression

$$v_i(r_m, r_c) = p_i w_i - (r_m + p_i r_c).$$

The level curves of v_i are linear with slope $-1/p_i$:

$$\{(r_m, r_c) : v_i(r_m, r_c) = C\} = \{(r_m,\ K - r_m/p_i) : r_m \in \mathbb{R}\} \tag{2}$$

where $K = w_i - C/p_i$.

We suppose the advertiser requires limited liability in the following sense. For advertiser specific constants $\mathrm{CPM}_i > 0$ and $\mathrm{CPC}_i > 0$, we assume the advertiser has strictly negative utility for any contract (r_m, r_c) with either $r_m > \mathrm{CPM}_i$ or $r_c > \mathrm{CPC}_i$; aside from this caveat, the advertiser is risk-neutral. In other words, advertisers effectively have a maximum amount they are willing to spend on clicks and impressions, but otherwise they are risk neutral.

The utility function of each advertiser A_i can be derived from four numbers: its value-per-click w_i, its estimated CTR p_i, and its price caps CPM_i and CPC_i. Equivalently, A_i's utility function is determined by the two contracts

$$\{(r_m^i, \mathrm{CPC}_i),\ (\mathrm{CPM}_i, r_c^i)\}$$

where $r_m^i = p_i(w_i - \mathrm{CPC}_i)$ and $r_c^i = w_i - \mathrm{CPM}_i/p_i$. These two IPC contracts lie on A_i's zero-utility level line; that is,

$$v_i(r_m^i, \mathrm{CPC}_i) = 0 \qquad v_i(\mathrm{CPM}_i, r_c^i) = 0.$$

Moreover, these contracts are the most "extreme" points on this zero-utility line (i.e., they push up against the price caps). Observe that w_i is the y-intercept of the line through these two contract points, and $p_i = \left(CPM_i - r_m^i\right) / \left(CPC_i - r_c^i\right)$ is the negative reciprocal of the slope of this line. Furthermore, the space of advertiser utility functions is parametrized by the set of contract pairs

$$U = \{(r_{m,1}, r_{c,1}), (r_{m,2}, r_{c,2}) \,|\, r_{m,1} \leq 0 < r_{m,2},\ r_{c,2} \leq 0 < r_{c,1}\}\,. \tag{3}$$

From the (risk-neutral) publisher's perspective, the utility of a contract (r_m, r_c) entered into with advertiser A_i is

$$v_i^{\mathcal{A}}(r_m, r_c) = r_m + p_i^A r_c$$

where p_i^A is the publisher's estimated CTR of the advertiser's ad. Figure 1 illustrates the contract preferences for an advertiser and a publisher.

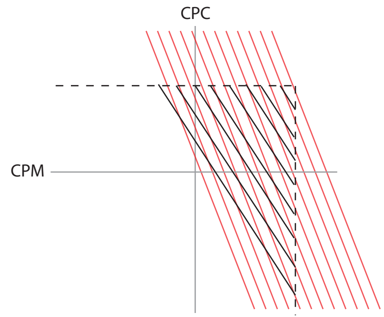

Fig. 1. The solid black lines indicate level curves of an advertiser's price preferences in the CPM-CPC price plane. The red lines indicate the corresponding level curves for a publisher whose estimated CTR for an ad is lower than the advertiser's own estimate.

3.2 Designing the Impression-Plus-Click Auction

Given the advertiser and publisher preferences outlined in Section 3.1, we next apply Theorem 1 to design a dominant strategy incentive compatible IPC auction for sponsored search. We start with two preliminary lemmas.

Lemma 1. *Assume the setting and notation of Mechanism 1, and the advertiser and publisher preferences of Section 3.1. Then the agents have consistent valuations with the publisher. Furthermore, letting* $\left\{(\tilde{r}_m^i, \widetilde{CPC}_i), (\widetilde{CPM}_i, \tilde{r}_c^i)\right\}$ *denote* A_i*'s reported preferences, we have*

$$R_i = \begin{cases} \widetilde{CPM}_i + p_i^A \tilde{r}_c^i & \text{if } p_i^A \leq \tilde{p}_i \\ \tilde{r}_m^i + p_i^A \widetilde{CPC}_i & \text{if } p_i^A \geq \tilde{p}_i \end{cases}$$

where $\tilde{p}_i = \left(\widetilde{CPM}_i - r_m^i\right) / \left(\widetilde{CPC}_i - r_c^i\right)$ *is* A_i*'s inferred (subjective) CTR.*

Proof. The level curve L_0 on which the advertiser has (true) zero utility is given by the line segment

$$\begin{aligned} L_0 &= \{(r_m, r_c) \,|\, r_m + p_i r_c = p_i w_i,\, r_m \leq \text{CPM}_i,\, r_c \leq \text{CPC}_i\} \\ &= \{(r_m, w_i - r_m/p_i) \,|\, p_i(w_i - \text{CPC}_i) \leq r_m \leq \text{CPM}_i\} \end{aligned}$$

and the set S_i on which the advertiser has non-negative utility is given by the points below L_0:

$$S_i = \{(r_m, r_c) \,|\, \exists\, (r_m^*, r_c^*) \in L_0 \text{ such that } r_m \leq r_m^* \text{ and } r_c \leq r_c^*\}\,.$$

If $(r_m, r_c) \in S_i \setminus L_0$ (i.e., if $v_i(r_m, r_c) > 0$), then there exists $(r_m^*, r_c^*) \in L_0$ such that either $r_m^* > r_m$ or $r_c^* > r_c$. In either case, $v_i^{\mathcal{A}}(r_m^*, r_c^*) > v_i^{\mathcal{A}}(r_m, r_c)$, and so A_i and the publisher have consistent valuations.

To compute R_i, we first assume agent A_i truthfully reports its preferences. Consistent valuations implies that the publisher achieves its maximum value, among contracts in S_i, on the set L_0 where the advertiser has zero utility. For $(r_m, r_c) \in L_0$,

$$\begin{aligned} v_i^{\mathcal{A}}(r_m, r_c) &= r_m + p_i^A r_c \\ &= r_m + (w_i - r_m/p_i) p_i^A \\ &= w_i p_i^A + r_m \left(1 - p_i^A / p_i\right). \end{aligned} \quad (4)$$

Now note that (4) is an increasing function of r_m for $p_i < p_i^A$, and a decreasing function of r_m for $p_i > p_i^A$. Consequently, the maximum is achieved at the endpoints of L_0. To extend to the case were A_i does not necessarily report truthfully, we need only replace A's actual preferences with its reported preferences, as indicated by the tildes. □

Lemma 2. *Assume the setting and notation of Lemma 1. Fix* $1 \le i \le N$ *and* $R_* \le R_i$. *Then for*

$$S = \left\{(r_m, r_c) \in C_i \,\middle|\, v_i^{\mathcal{A}}(r_m, r_c) \ge R_*\right\}$$

we have

$$\arg\max_S \tilde{v}_i(r_m, r_c) = \begin{cases} \left(\widetilde{CPM}_i, (R_* - \widetilde{CPM}_i)/p_i^A\right) & \text{if } p_i^A < \tilde{p}_i \\ \left(R_* - p_i^A \widetilde{CPC}_i, \widetilde{CPC}_i\right) & \text{if } p_i^A > \tilde{p}_i \\ T & \text{if } p_i^A = \tilde{p}_i \end{cases}$$

where

$$T = \left\{ \left(r_m, (R_* - r_m)/p_i^A\right) \,\middle|\, R_* - p_i^A \widetilde{CPC}_i \le r_m \le \widetilde{CPM}_i \right\}.$$

Proof. First note that since $R_* \le R_i$, $\max_S \tilde{v}_i \ge 0$. Now, the level curve L_A on which $v_i^{\mathcal{A}}(r_m, r_c) = R_*$ is given by

$$\begin{aligned} L_A &= \left\{(r_m, r_c) \,\middle|\, r_m + p_i^A r_c = R_*\right\} \\ &= \left\{ \left(r_m,\ (R_* - r_m)/p_i^A\right) \,\middle|\, r_m \in \mathbb{R}\right\}. \end{aligned}$$

Furthermore, $v_i^{\mathcal{A}}(r_m, r_c) > R_*$ if and only if (r_m, r_c) lies above this line. That is, $v_i^{\mathcal{A}}(r_m, r_c) > R_*$ if and only if there exists a contract $(r_m^*, r_c^*) \in L_A$ such that either $r_m \ge r_m^*$ and $r_c > r_c^*$, or $r_m > r_m^*$ and $r_c \ge r_c^*$. In either case, $\tilde{v}_i(r_m^*, r_c^*) > \tilde{v}_i(r_m, r_c)$ and so $\arg\max_S \tilde{v}_i \subseteq L_A$. Since $\max_S \tilde{v}_i \ge 0$, we can further restrict ourselves to the set

$$\begin{aligned} T &= L_A \cap (-\infty, \widetilde{\mathrm{CPM}}_i] \times (-\infty, \widetilde{\mathrm{CPC}}_i] \\ &= \left\{ \left(r_m, (R_* - r_m)/p_i^A\right) \,\middle|\, R_* - p_i^A \widetilde{\mathrm{CPC}}_i \le r_m \le \widetilde{\mathrm{CPM}}_i \right\}. \end{aligned}$$

For $(r_m, r_c) \in T$, and $\tilde{w}_i$ indicating A_i's inferred value per click, we have

$$\begin{aligned} \tilde{v}_i(r_m, r_c) &= \tilde{w}_i \tilde{p}_i - [r_m + \tilde{p}_i r_c] \\ &= \tilde{w}_i \tilde{p}_i - \left[r_m + (R_* - r_m)\tilde{p}_i/p_i^A\right] \\ &= \tilde{w}_i \tilde{p}_i - R_* \tilde{p}_i/p_i^A + r_m \left(\tilde{p}_i/p_i^A - 1\right). \end{aligned} \tag{5}$$

The result now follows by noting that (5) is increasing in r_m for $p_i^A < \tilde{p}_i$, decreasing for $p_i^A > \tilde{p}_i$, and constant for $p_i^A = \tilde{p}_i$. □

Together with Lemmas 1 and 2, the general contract auction of Mechanism 1 leads to the impression-plus-click auction described by Mechanism 2. First, each advertiser submits two contracts—ostensibly specifying its entire utility function. The publisher then computes its own utility for each of these $2N$ contracts, and the winner of the auction is the agent who submitted the contract with the highest value to the publisher. The "second-highest value" is the value of the best contract (again from the publisher's perspective) among those submitted by the losing bidders. To determine the actual contract entered into, we consider two cases. If the highest value contract has higher CPM than the winner's other bid, then the final contract is determined by decreasing the CPC on the highest value contract until the publisher's value for that contract is equal to the second highest value. Analogously, if the highest value contract has lower CPM than the winner's other bid, the final contract is determined by decreasing the CPM of the highest value contract. Determination of the final contract is illustrated in Figure 2.

Mechanism 2 An Impression-Plus-Click Auction

1: Advertisers $A_1, \ldots, A_N$ each report their valuation functions, encoded by the pair of extremal contracts as described in Section 3.1:

$$\tilde{v}_i = \left\{ \left(r^i_{m,1}, r^i_{c,1} \right), \; \left(r^i_{m,2}, r^i_{c,2} \right) \right\}$$

where $r^i_{m,1} \leq 0 < r^i_{m,2}$ and $r^i_{c,2} \leq 0 < r^i_{c,1}$.

2: For each report $\tilde{v}_i$ define

$$\begin{aligned} R_i &= \max\left(v^{\mathcal{A}}_i \left(r^i_{m,1}, r^i_{c,1} \right), \; v^{\mathcal{A}}_i \left(r^i_{m,2}, r^i_{c,2} \right) \right) \\ &= \max\left(r^i_{m,1} + r^i_{c,1} p^A_i, \; r^i_{m,2} + r^i_{c,2} p^A_i \right). \end{aligned}$$

3: Fix h so that

$$R_{h(1)} \geq R_{h(2)} \geq \cdots \geq R_{h(N)}.$$

The publisher enters into a contract with agent $A_{h(1)}$. The final contract c^* is determined as follows:

$$c^* = \begin{cases} \left(r^{h(1)}_{m,2}, \; \left(R_{h(2)} - r^{h(1)}_{m,2} \right) \big/ \, p^A_{h(1)} \right) & \text{if } R_{h(1)} = v^A_{h(1)} \left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2} \right) \\ \left(R_{h(2)} - p^A_{h(1)} r^{h(1)}_{c,1}, \; r^{h(1)}_{c,1} \right) & \text{otherwise} \end{cases}$$

Theorem 2. *Consider the setting and notation of Mechanism 2 with the advertiser and publisher preferences of Section 3.1. Then*

1. $c^* \leq \left(r^{h(1)}_{m,1}, r^{h(1)}_{c,1} \right)$ *or* $c^* \leq \left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2} \right)$, *where the inequalities hold coordinate-wise.*
2. *The mechanism is dominant strategy incentive compatible. That is, it is a dominant strategy for each advertiser A_i to truthfully report*

$$\left\{ \left(r^i_m, CPC_i \right), \; \left(CPM_i, r^i_c \right) \right\}.$$

4 The Impression-Or-Click Pricing Model

With impression-plus-click pricing, advertisers pay publishers for each impression, and then pay an additional amount if their ad is clicked. The hybrid sponsored search auction of Goel & Munagala [7] can be thought of as *impression-or-click* (IOC) pricing. That is, the final selected contract is guaranteed to be either pure per-impression or pure per-click, but it is not known which it will be until all bids have been submitted.

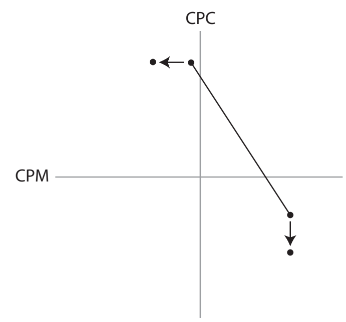

Fig. 2. Settling the impression-plus-click auction. The final contract is determined by decreasing either the CPM or the CPC (but not both) of the highest value contract to the publisher.

The hybrid auction, as shown below, is equivalent to a special case of the general contract auction with the contract spaces restricted to the axes of the CPM-CPC plane:

$$C_i = \{(r_m, 0) \mid r_m \in \mathbb{R}\} \cup \{(0, r_c) \mid r_c \in \mathbb{R}\}. \quad (6)$$

Suppose both advertisers and publishers are risk neutral. As before, let p_i denote advertiser A_i's subjective click-through rate, let p_i^A denote the publisher's estimated click-through rate for an impression awarded to A_i, and let w_i denote A_i's value for a click.[1] Then A_i has zero utility for the two contracts $(p_i w_i, 0)$ and $(0, w_i)$. By the assumption of risk neutrality, these two contracts completely determine A_i's preferences over all contracts. Hence, A_i can communicate its preferences by reporting the two numbers $\text{CPM}_i = p_i w_i$ and $\text{CPC}_i = w_i$, corresponding to the maximum it is willing to pay for a per-impression and a per-click contract, respectively. The resulting IOC auction is outlined in Mechanism 3, and details of its derivation are left to the Appendix.[2]

Mechanism 3 An Impression-Or-Click Auction (Goel & Munagala)

1: Advertisers $A_1, \ldots, A_N$ each report their valuation functions, encoded by the constants $\widetilde{CPM}_i, \widetilde{\text{CPC}}_i > 0$.

2: For each report, define

$$R_i = \max\left(\text{CPM}_i,\ p_i^A \text{CPC}_i\right).$$

3: Fix h so that

$$R_{h(1)} \geq R_{h(2)} \geq \cdots \geq R_{h(N)}.$$

Then the publisher enters into a contract with agent $A_{h(1)}$. The final contract c^* is determined as follows:

$$c^* = \begin{cases} (0, R_*/p_i^A) & \text{if } R_{h(1)} = p_{h(1)}^A \text{CPC}_{h(1)} \\ (R_*, 0) & \text{otherwise} \end{cases}$$

---

[1] As before, without loss of generality we may assume advertisers have no inherent value for impressions that are not clicked.

[2] The hybrid auction [7] requires only that $\widetilde{CPM}_i, \widetilde{\text{CPC}}_i \geq 0$; for simplicity, however, we restrict to strict inequality.

Although the hybrid and general contract auctions are equivalent when advertiser preferences are restricted to the CPM-CPC axis, they may lead to different outcomes when preferences are defined over the entire plane. Consider the IPC auction setting of Section 3, where we now assume that $\text{CPC}_i = w_i$ and $\text{CPM}_i = w_i p_i$. That is, the most advertiser A_i is willing to pay per click or per impression is, respectively, its true per click value w_i and its true per impression value $w_i p_i$. In particular, A_i will not pay more than w_i per click even if it is compensated via negative per-impression payments. In this case, the two extremal contracts that define A_i's utility function over the CPM-CPC plane are $(\text{CPM}_i, 0)$ and $(0, \text{CPC}_i)$. With such a preference profile, we show that advertisers prefer the IPC auction over the IOC auction, and publishers are ambivalent between the two.

In both the IOC and IPC auctions, it is dominant to truthfully reveal ones' preferences: In the IOC auction advertisers report their maximum per-impression and per-click payments CPM_i and CPC_i; in the IPC auction they report their pair of extremal contracts $\{(\text{CPM}_i, 0), (0, \text{CPC}_i)\}$. From the publisher's perspective, for each agent A_i, R_i is the same in both auctions. Consequently, the winner of the auction is the same under either mechanism, and moreover, the expected (subjective) revenue R_* of the publisher is also the same. The publisher is thus ambivalent between the IOC and IPC auction designs.

From the advertisers' view, however, the situation is quite different. Specifically, let c^*_{IPC} and c^*_{IOC} denote the final contract entered into by the winner $A_{h(1)}$ under each mechanism. Then

$$v_{h(1)}(c^*_{\text{IPC}}) = \max_{Q_1} v_{h(1)} \qquad v_{h(1)}(c^*_{\text{IOC}}) = \max_{Q_2} v_{h(2)}$$

where

$$Q_1 = \left\{ (r_m, r_c) \in \mathbb{R}^2 \;\middle|\; v^A_{h(1)}(r_m, r_c) \geq R_* \right\}$$
$$Q_2 = \left\{ (r_m, r_c) \in \mathbb{R}^2 \;\middle|\; v^A_{h(1)}(r_m, r_c) \geq R_*,\ \min(r_m, r_c) = 0 \right\}$$

That is, the IPC contract is optimized over the entire plane, whereas the IOC contract is optimized only over the axes. In particular, $v_{h(1)}(c^*_{\text{IPC}}) \geq v_{h(1)}(c^*_{\text{IOC}})$. Since $v^A_i(c^*_{\text{IPC}}) = v^A_i(c^*_{\text{IOC}})$, the line drawn between these two contracts has slope $-1/p^A_i$ (as shown in Section 3.1). Furthermore, since $v_i(c^*_{\text{IPC}}) = v_i(c^*_{\text{IOC}})$ if and only if the line between the contracts has slope $-1/p_i$, we have $v_{h(1)}(c^*_{\text{IPC}}) > v_{h(1)}(c^*_{\text{IOC}})$ provided that $p^A_i \neq p_i$. Hence, in this setting, advertisers typically prefer the IPC over the IOC auction.

The distinction between the IPC and IOC settlement mechanisms is illustrated in Figure 3. When $p^A_{h(1)} < p_{h(1)}$, the publisher prefers (under both mechanisms) the winning advertiser's pure per-impression bid $\text{CPM}_{h(1)}$ over its pure per-click bid $\text{CPC}_{h(1)}$. In this case, the final IOC contract is a pure per-impression contract, where the per-impression payment is reduced from $\text{CPM}_{h(1)}$ to an amount such that the ultimate value of the contract to the publisher is R_*. In contrast, the final IPC contract has the advertiser still paying $\text{CPM}_{h(1)}$ per impression, but a "discount" is given to the advertiser via negative click payments (i.e., the publisher pays the advertiser for each click). This negative click payment is calculated so that the final value of the contract to the publisher is again R_*. The final contract in either auction lies on the R_* level curve of the publisher: In the IOC auction, the pure-impression contract $\text{CPM}_{h(1)}$ is moved left along the CPM axis until hitting this level curve;

in the IPC auction, the final contract is arrived at by moving the pure-impression contract down parallel to the CPC axis.[3]

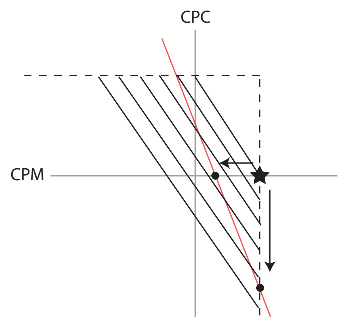

Fig. 3. The IOC and the IPC auctions arrive at a final contract by moving along different axes in the CPM-CPC plane. The star indicates a winning pure per-impression bid, the red line is the publisher's R_* level line, and the two dots indicate the final contracts under each auction mechanism.

5 Discussion

General contract auctions facilitate transactions when parties have conflicting information, or when they simply have different inherent value for the specific terms of a contract. Such a situation is common in traditional business negotiations, and, at least implicitly, contracts in the offline world often balance the same tradeoffs encapsulated explicitly by impression-plus-click auctions. For example, with book publication, authors typically receive a one-time advance plus royalty fees (i.e., a percentage of total sales revenue). Thus, authors confident in the future success of their book should be willing to trade a smaller advance for larger royalties. A similar tradeoff occurs with insurance premiums and deductibles: A driver who thinks he is unlikely to get into an accident should be willing to accept relatively high deductibles in exchange for relatively low premiums. Executives also face a similar situation when deciding between guaranteed salaries and performance-based bonuses.

As with impression-plus-click sponsored search pricing, negative payments are potentially applicable in the offline world as well. For example, one may be willing to pay a high premium for earthquake insurance in exchange for a "disaster bonus." Analogously, an author may be willing to pay to have their book published in exchange for an especially high percentage of the revenue.

In such instances where parties bargain between deterministic and stochastic payments, a design similar to the impression-plus-click auction may prove useful.

[3] When $p^A_{h(1)} > p_{h(1)}$ the situation is reversed: The pure click contract is preferred by the auctioneer; the IOC auction selects the final contract by moving down along the CPC axis until hitting the publisher's R_* level line; and the IPC auction hits this level line by moving left parallel to the CPM axis, corresponding to negative impression payments.

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A Technical Details and Proofs

Proof of Theorem 2.

Proof. First observe that both possible final contracts have value $R_{h(2)}$ to the publisher. Consequently,

$$v^A_{h(1)}(c^*) \leq R_{h(1)} = \max\left(v^A_{h(1)}\left(r^{h(1)}_{m,1}, r^{h(1)}_{c,1}\right), v^A_{h(1)}\left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2}\right)\right).$$

If $R_{h(1)} = v^A_{h(1)}\left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2}\right)$, then we have

$$v^A_{h(1)}\left(r^{h(1)}_{m,2}, c^*_c\right) \leq v^A_{h(1)}\left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2}\right)$$

where c^*_c is the cost per click in the final contract c^*. By monotonicity of $v^A_{h(1)}$, $c^*_c \leq r^{h(1)}_{c,2}$. When $R_{h(1)} \neq v^A_{h(1)}\left(r^{h(1)}_{m,2}, r^{h(1)}_{c,2}\right)$, an analogous argument shows that $c^* \leq \left(r^{h(1)}_{m,1}, r^{h(1)}_{c,1}\right)$.

To establish incentive compatibility, note that by Lemma 1 the variable R_i as defined in Mechanism 2 indeed corresponds to the variable of the same name in the general contract auction described in Mechanism 1 (i.e., R_i is the value of the best contract—from the perspective of the publisher—for which the advertiser reports to have non-negative utility). Moreover, Lemma 1 shows that if R_i is achieved on A_i's "rightmost" contract (i.e., its contract with higher CPM), then $p^A_i \leq \tilde{p}_i$. In this case, by Lemma 2, A_i reportedly prefers the contract ultimately chosen by the mechanism. The analogous result holds when R_i is achieved on the "leftmost" contract. □

To describe the impression-or-click auction, where the contract spaces are given by (6), we derive analogs of Lemmas 1 and 2.

Lemma 3. *Consider the setting and notation of Mechanism 1 with contract spaces given by* (6)*, and risk-neutral advertisers and publishers. Then the agents have consistent valuations with the publisher. Furthermore, letting* $\widetilde{CPM}_i$ *and* $\widetilde{CPC}_i$ *denote* A_i*'s reported preferences, and* $\tilde{p}_i = \widetilde{CPM}_i/\widetilde{CPC}_i$ *its inferred (subjective) click-through rate, we have*

$$R_i = \begin{cases} \widetilde{CPM}_i & \text{if } p^A_i \leq \tilde{p}_i \\ p^A_i \widetilde{CPC}_i & \text{if } p^A_i \geq \tilde{p}_i \end{cases}$$

Proof. To show consistency, observe that A_i has strictly positive value if and only if the contract is in the set $H_1 \cup H_2$ where

$$H_1 = \{(r, 0) \mid r < \text{CPM}_i\} \qquad H_2 = \{(0, r) \mid r < \text{CPC}_i\}.$$

Now, $v_i((\text{CPC}_i, 0)) = v_i((0, \text{CPM}_i)) = 0$, and $\max_{H_1} v^A_i(c) < v^A_i((\text{CPM}_i, 0))$ and $\max_{H_2} v^A_i(c) < v^A_i((0, \text{CPC}_i))$. From this consistency follows.

To compute R_i, we first assume A_i truthfully reports its preferences. By consistency, R_i is attained on the set $\{(\text{CPM}_i, 0), (0, \text{CPC}_i)\}$, and in particular, $v^A_i((\text{CPM}_i, 0)) = \text{CPM}_i$ and $v^A_i((0, \text{CPC}_i)) = p^A_i \text{CPC}_i$. Thus,

$$R_i = \begin{cases} \text{CPM}_i & \text{if } p^A_i \leq p_i \\ p^A_i \text{CPC}_i & \text{if } p^A_i \geq p_i \end{cases}$$

For the general case, where A_i does not necessarily report truthfully, we replace its true values CPM_i and CPC_i with its reported values $\widetilde{\text{CPM}}_i$ and $\widetilde{\text{CPC}}_i$, and use the derived click-though rate $\tilde{p}_i$ instead of A_i's own true (subjective) click-through rate p_i. □

Lemma 4. *Consider the setting and notation of Lemma 3. Fix $1 \leq i \leq N$ and $R_* \leq R_i$. Then for*

$$S = \left\{(r_m, r_c) \in C_i \,|\, v_i^A(r_m, r_c) \geq R_*\right\}$$

we have

$$\arg\max_S \tilde{v}_i(r_m, r_c) = \begin{cases} (R_*, 0) & \text{if } p_i^A < \tilde{p}_i \\ (0, R_*/p_i^A) & \text{if } p_i^A > \tilde{p}_i \\ \{(R_*, 0), (0, R_*/p_i^A)\} & \text{if } p_i^A = \tilde{p}_i \end{cases}$$

where $\tilde{p}_i = \widetilde{CPM}_i / \widetilde{CPC}_i$ is inferred from agent A_i's report.

Proof. Observe that S can be re-written as

$$S = \{(r, 0) \,|\, r \geq R_*\} \cup \{(0, r) \,|\, r \geq R_*/p_i^A\}.$$

Then the max of $\tilde{v}_i$ over S is attained on the boundary points $\{(R_*, 0), (0, R_*/p_i^A)\}$. The result follows by observing that $\tilde{v}_i((R_*, 0)) = \widetilde{\text{CPC}}_i - R_*$ and $\tilde{v}_i\left((0, R_*/p_i^A)\right) = \widetilde{\text{CPC}}_i - R_* p_i / p_i^A$. □

Analogous to the proof of Theorem 2, Lemmas 3 and 4 together with Mechanism 1 yield the hybrid auction of Goel & Munagala [7] described in Mechanism 3.