Stability and Incentive Compatibility in a Kernel-Based Combinatorial Auction

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Abstract

We present the design and analysis of an approximately incentive-compatible combinatorial auction. In just a single run, the auction is able to extract enough value information from bidders to compute approximate truth-inducing payments. This stands in contrast to current auction designs that need to repeat the allocation computation as many times as there are bidders to achieve incentive compatibility. The auction is formulated as a kernel method, which allows for flexibility in choosing the price structure via a kernel function. Our main result characterizes the extent to which our auction is incentive-compatible in terms of the complexity of the chosen kernel function. Our analysis of the auction's properties is based on novel insights connecting the notion of stability in statistical learning theory to that of universal competitive equilibrium in the auction literature.

Introduction

The purpose of an auction is typically to determine an efficient allocation of resources. To achieve this, the auction must not only compute the value maximizing allocation, but also incentivize buyers to truthfully report their values so that the maximization is performed over the right objective. In the process of computing an efficient allocation, auctions—especially iterative auctions—often derive clearing prices that balance supply and demand. However, when one deals with combinatorial auctions over multiple items, these prices may not induce truthful reporting. Instead, one must perform further computation to obtain truth-inducing payments. Due to a variety of appealing properties, the Vickrey-Clarke-Groves (VCG) payment scheme is often the method of choice to achieve incentive compatibility (Clarke 1971; Groves 1979; Vickrey 1961).

This situation has led to much research into combinatorial auctions that derive VCG payments as a by-product of the allocation and clearing process, and into characterizations of the conditions under which prices can match VCG payments. Bikhchandani and Ostroy (2002) demonstrate that the latter is the case when valuations satisfy a substitutes condition. Ausubel (2006) and de Vries et al. (2007) develop linear and nonlinear price iterative auctions that compute VCG payments under this substitutes condition. Mishra and Parkes (2007) develop an ascending-price iterative auction for general valuations that terminates with sufficient information to compute VCG payments.

The sealed-bid VCG mechanism itself can be run to compute incentivizing payments. This requires solving the allocation problem once with all buyers present and n more times each with one agent removed, where n is the number of agents, for an n-fold increase in computation over simple efficient allocation. In fact, all the auctions mentioned above also introduce an n-fold increase in computation, either implicitly or explicitly, by requiring multiple price paths or increasing the computation in each round—their advantages lie not in their computational savings but elsewhere (e.g., ascending prices, incremental elicitation). The question of how much increase in computation is needed to compute VCG payments goes back to the work of Nisan and Ronen (1999), who posed it specifically in the context of routing on a network.

In this work, we study the extent to which clearing prices computed by an auction can capture the information needed to derive VCG payments after just a *single* run of the allocation process. We adapt the original formulation of the combinatorial allocation problem given by Bikhchandani and Ostroy (2002) in two ways, drawing on ideas from kernel methods in machine learning. First, following Lahaie (2009), we introduce some flexibility in the choice of price structure via a kernel function that specifies how bundles are priced. Second, we introduce a penalty term on the magnitude of the clearing prices, akin to a regularization term in machine learning. This formulation of the allocation problem can form the basis of single-shot or iterative combinatorial auctions, using standard techniques from the literature (Bikhchandani et al. 2001).

Our main result characterizes how well VCG payments can be approximated using the price information computed by our auction formulation, in terms of the price structure and penalty term. Our analysis draws a connection between the notion of *stability* (Bousquet and Elisseeff 2002) in statistical learning theory and the notion of a *universal competitive equilibrium* (Mishra and Parkes 2007) in the auction literature. Informally, stability ensures that a learning algorithm is not too sensitive to any particular input in the training data, so that the algorithm can therefore avoid overfitting and generalize well. Under a universal competitive equilibrium, the clearing prices remain clearing even if an agent is removed; thus, the prices are not sensitive to the presence of any one agent. This does not mean that we achieve incentive-compatibility with such prices, but that they capture an agent's marginal contribution to the economy, which is the information needed to compute VCG payments.

The remainder of the paper is organized as follows. We first present the model, explaining the distinction between prices and payments in auctions. We then provide our kernel-based auction formulation. The following section contains our results, connecting the concepts of stability, universal competitive equilibrium, VCG payments, and incentive compatibility. We conclude with a discussion of our bounds and avenues for future research.

The Model

There are n buyers and m distinct indivisible items held by a single seller. A *bundle* is a subset of the items. We associate each bundle with its indicator vector, and denote the set of bundles by $X = \{0,1\}^m$. We write $x \leq x'$ to denote that bundle x is contained in bundle x' (the inequality is understood component-wise).

Buyer i has a value function $v_i : X \rightarrow \mathbf{R}_+$ denoting how much it is willing to pay for each bundle. For the sake of simplicity, we assume that each buyer is *single-minded*, meaning that there is a bundle x_i such that v_i is entirely determined by the bundle-value pair $(x_i, v_i(x_i))$ as follows:

$$v_i(x) = \begin{cases} v_i(x_i) & \text{if } x \geq x_i \\ 0 & \text{otherwise} \end{cases}$$

with the requirement that $v_i(\emptyset) = 0$. In words, buyer i would like to acquire all the items in bundle x_i, but is not interested in any others.[1]

A *selection* of buyers is simply a subset $I \subseteq \{1, \dots, n\}$. A selection if *feasible* if $\sum_{i \in I} x_i \leq \mathbf{1}$; this means that the desired bundles of the buyers in the selection do not intersect. We denote the set of feasible selections by $\mathcal{F}$. A selection I is *efficient* if $I \in \arg\max_{I' \in \mathcal{F}} \sum_{i \in I'} v_i(x_i)$. In this paper we will typically denote the allocation of items corresponding to a selection by a vector of bundles $y = (y_1, \dots, y_n)$, where y_i is the bundle acquired by buyer i; we have $y_i = x_i$ if $i \in I$ and $y_i = \mathbf{0}$ otherwise. Together with an efficient selection, we wish to determine transfers from the buyers to the auctioneer. Transfers can serve different purposes.

Prices In the first instance, transfers are specified as *prices* $p : X \rightarrow \mathbf{R}$ defined over bundles; we will assume that prices are normalized so that $p(\mathbf{0}) = 0$. The purpose of prices is to balance supply and demand. Prices are *competitive* with respect to a selection I if the following conditions hold:

$$\begin{array}{ll} v_i(x_i) - p(x_i) \geq 0 & \text{for } i \in I \\ v_i(x_i) - p(x_i) \leq 0 & \text{for } i \notin I \\ \sum_{i \in I} p(x_i) \geq \sum_{i \in I'} p(x_i) & \text{for all } I' \in \mathcal{F} \end{array}$$

This means that each buyer in the selection willingly acquires its associated bundle, the other buyers willingly receive nothing, and the selection maximizes the seller's revenue. In this sense, the prices balance supply and demand, assuming the agents act as pure price-takers. If prices p are competitive with respect to a selection I, we say that they *support* the selection and the associated allocation y.

It is well-known that if prices p are competitive with respect to a selection, then the selection is efficient (Bikhchandani and Ostroy 2002). It is also known that with single-minded bidders, competitive prices exist to support any efficient selection (Lahaie and Parkes 2009); however, existence may not hold if we impose a specific structure on the prices, such as linearity.[2] Part of our goal in designing our auction is to allow for flexibility in the price structure, in order to study the interplay between price structure and incentives.

Payments In the second instance, transfers are specified as *payments* $q = (q_1, \dots, q_n) \in \mathbf{R}^n$ from the buyers to the seller. The purpose of payments is to align the buyers' incentives with that of efficiency. A rational buyer will realize that both the selection and its payment in an auction will depend on the value function it reports, and will therefore choose its report to maximize its own surplus (i.e., it may lie about its value function), where surplus refers to value minus price. We seek a payment rule such that an agent's potential gain from misreporting its value is small.

Formally, let (y, q) be the allocation and payments computed by the auction when buyer h reports its value function truthfully, holding the other buyers' values fixed, and let (y', q') be the allocation and payments computed when it reports some other value function. The auction is ϵ-*incentive compatible* (in dominant strategies), where $\epsilon \geq 0$, if it is always the case that

$$v_h(y_h) - q_h + \epsilon \geq v_h(y'_h) - q'_h.$$

If ϵ is not explicitly mentioned, it is implied that $\epsilon = 0$. A natural approach would be to charge $q_i = p(y_i)$ for each buyer i, but with general or even single-minded valuations this may not be ϵ-incentive compatible for any small ϵ (Ausubel and Milgrom 2006), no matter what competitive prices are chosen. A standard way to achieve incentive-compatibility is to use the Vickrey-Clarke-Groves payment scheme, to be defined in the next section.

The Auction

Kernels To achieve flexibility in the price structure, we follow Lahaie (2009) and draw on ideas from kernel methods in machine learning. To compute nonlinear competitive prices, we view these as linear prices in a higher-dimensional space to which we map the bundles via a mapping $\phi : X \rightarrow \mathbf{R}^M$, where usually $M \gg m$. Given

[1] All our results extend to arbitrary valuations over bundles, but the auction formulation and analysis would be needlessly complex. Only Lemma 1 would have to be generalized, the remaining proofs would remain unchanged.

[2] When generalizing our results to arbitrary valuations, we may need personalized prices to ensure the existence of competitive prices (i.e., we may need $p_i(x) \neq p_j(x)$ for two distinct buyers i and j). This can be handled in our model by considering bundle-agent pairs (x, i) instead of just bundles, and redefining valuations so that an agent's value for a bundle-value pair is zero unless the bundle is earmarked for that agent.

$p \in \mathbf{R}^M$, the price of bundle x is then $\langle p, \phi(x) \rangle$. These linear prices in $\mathbf{R^M}$ translate into nonlinear prices over the original bundle space X. (With a slight abuse of notation, we will often write $p(x)$ to mean $\langle p, \phi(x) \rangle$, keeping ϕ implicit.) The issue here is that the dimension M may be very large (even infinite), so it may be infeasible to work explicitly with prices in $\mathbf{R}^M$.

The trick used to address this in kernel methods is to formulate the relevant problem (e.g., classification, regression) as a mathematical program that relies only on the inner products $\langle \phi(x_i), \phi(x_j) \rangle$. What makes this practical is that, for many kinds of mappings, the inner products can be efficiently evaluated in time that does not depend on M. The inner products are given via a *kernel function* k over bundle pairs, defined as $k(x_1, x_2) = \langle \phi(x_1), \phi(x_2) \rangle$. We will assume that the mapping ϕ is centered, meaning $\phi(\mathbf{0}) = \mathbf{0}$; this ensures that prices are normalized. If this is not the case, we can simply work with the alternate mapping $\phi' = \phi - \phi(\mathbf{0})$ and its associated kernel function.

We introduce two different kernels for the purpose of discussing our bounds, corresponding to two extremes. The *linear kernel* is defined as $k(x, x') = \langle x, x' \rangle$, corresponding to the mapping $\phi(x) = x$. This gives us the simplest possible price structure of linear (i.e., item) prices. At the other extreme we have the *identity* kernel: $k(x, x') = 1$ if $x = x'$ and 0 otherwise. This corresponds to a mapping to $\mathbf{R}^{2^m}$ where we have a dimension for each bundle; a bundle gets sent to the unit vector that has a 1 in its associated dimension. The identity kernel gives the most complex possible price structure: each bundle is independently priced. As mentioned, competitive prices always exist when one uses the identity kernel. We stress that these kernels are for illustration purposes only. In applications one would rely on something more practical, but we do not address the question of designing auction kernels here because this is necessarily a domain-specific exercise.

Formulation With a mapping ϕ and associated kernel function k at hand, we formulate the problem of finding an efficient outcome via the following quadratic program (P):

$$\begin{aligned} \max_{\alpha \geq 0, \bar{\alpha} \geq 0} \quad & \sum_{i=1}^{n} \alpha_i v_i(x_i) \\ & - \frac{1}{2\lambda} \left\| \sum_{i=1}^{n} \left(\alpha_i - \sum_{I \ni i} \bar{\alpha}_I \right) \phi(x_i) \right\|^2 && (1) \\ \text{s.t.} \quad & \alpha_i \leq 1 && i = 1, \ldots, n \\ & \sum_{I \in \mathcal{F}} \bar{\alpha}_I \leq 1 && I \in \mathcal{F} \end{aligned}$$

(Throughout the paper, $\|\cdot\|$ refers to the Euclidean norm.) Here variable α_i controls whether buyer i obtains its bundle x_i, and $\bar{\alpha}_I$ controls whether the seller makes selection I. Rather than imposing a hard constraint that the outcome be feasible, we introduce a penalty term (1) in the objective. By making λ small enough, the solution will be feasible to within any desired tolerance; see Chapter 4.2 of Bertsekas (1999). The use of a penalty term rather than a constraint allows us to solve the program using only inner product information because the squared norm evaluates to $\beta' K \beta$, where $\beta_i = (\alpha_i - \sum_{I \ni i} \bar{\alpha}_i)$ and K is an $n \times n$ matrix defined by $K(i, j) = k(x_i, x_j)$. If the solution is integer, the program has identified an efficient outcome. However, an integer solution will be guaranteed only if the price structure is sufficiently complex to allow it (Lahaie 2009). With the identity kernel, for instance, there is an integer solution (Bikhchandani and Ostroy 2002).

The primal objective is convex because K is positive semi-definite by Mercer's theorem (Schölkopf and Smola 2001), and the Slater condition holds almost trivially. Thus strong duality holds between (P) and its dual (Bertsekas 1999). The dual of the quadratic program is the following, denoted (D):

$$\begin{aligned} \min_{\pi \geq 0, \bar{\pi} \geq 0, p} \quad & \sum_{i=1}^{n} \pi_i + \bar{\pi} + \frac{\lambda}{2} \|p\|^2 \\ \text{s.t.} \quad & \pi_i \geq v_i(x_i) - \langle p, \phi(x_i) \rangle \quad i = 1, \ldots, n \\ & \bar{\pi} \geq \left\langle p, \sum_{i \in I} \phi(x_i) \right\rangle \qquad I \in \mathcal{F} \end{aligned}$$

We see that this program computes competitive prices. By complementary slackness, if $\alpha_i = 1$, then agent i obtains $y_i = x_i$ and $\pi_i = v_i(x_i) - \langle p, \phi(x_i) \rangle \geq 0$; whereas if $\alpha_i = 0$, then $y_i = \emptyset$ and $0 \geq v_i(x_i) - \langle p, \phi(x_i) \rangle$. Similarly, the selection corresponding to $\bar{\alpha}$ maximizes the seller's revenue. Of course, if the primal does not find an integer solution, then the dual solution is not useful. By the relationship between the primal and dual solutions, the prices p can be evaluated as

$$p(x) = \frac{1}{\lambda} \sum_{i=1}^{n} \left(\alpha_i - \sum_{I \ni i} \bar{\alpha}_I \right) k(x_i, x). \qquad (2)$$

This provides a sparse representation of the prices p implicitly computed in the dual once one has solved the primal.

The dual program has a penalty term on $\|p\|$ in the objective. In machine learning, the purpose of this term is to achieve some *stability* in the solution via large λ, so that it does not depend too much on any particular point in the training set; this yields good generalization ability (Bousquet and Elisseeff 2002). In our setting, the penalty term has deep connections to incentive compatibility. The first hint of this arises in the assignment problem, where each buyer acquires only a single item. There linear competitive prices exist, and the (unique) minimal competitive prices (i.e., those that minimize $\|p\|$) happen to coincide with VCG payments, thus achieving incentive compatibility (Leonard 1983). In this work, we study these connections for more general price structures.

From now on, we assume that k has been appropriately chosen so that (P) has an integer solution, and that λ has been chosen small enough so that the solution is feasible to within a small tolerance. We will address how to choose these in a principled way in the final section, once we understand their impact on incentive-compatibility. We consider an auction that proceeds by solving (P) to identify an efficient selection I corresponding to an allocation y, and then charges payments. Recall that once y is obtained, the auction has implicitly computed competitive prices p via (2).

We consider two different payment schemes. To define these, let y^{-h} be the allocation computed by the auction when buyer h is removed, and let $\tilde{y}^{-h}$ be a revenue-maximizing allocation with respect to prices p when buyer h is removed. Then the payments from buyer h are defined as follows:

$$q_h = \sum_{i \neq h} v_i(y_i^{-h}) - \sum_{i \neq h} v_i(y_i)$$

$$r_h = \sum_{i \neq h} p(\tilde{y}_i^{-h}) - \sum_{i \neq h} p(y_i)$$

Payments q are the familiar VCG payments. Because the auction implements an efficient selection, it follows from standard VCG arguments that it would be incentive-compatible if it charged payments q. Note that to compute payments q, one would have to re-run the auction n times to identify the allocations y^{-h} for each buyer h, which can be prohibitive.

We will see that payments r approximate VCG payments q. To compute payments r, one needs to run revenue maximization over all agents besides h to obtain the left-hand term in r_h, for all agents h. Note that this only requires the prices p as an input, not the agent values. In a single-shot auction, computing r in this manner would be as computationally intensive as re-running the auction n times in the worst case, but could be substantially easier if we happen to have a simple price structure. The crucial point is that each payment r_h is solely a function of the prices derived after solving (P) once. This distinguishes our approach from all the auctions mentioned in the introduction, which implicitly solve their underlying allocation problem $n+1$ times to obtain VCG payments.

Computation Although the purpose of this paper is to analyze the properties of solutions to (P), let us discuss briefly how the program could be solved in practice—different ways of solving the program would correspond to different ways of running the auction (e.g., as a single-shot or iterative auction). The kernel trick has made the number of constraints manageable, but the number of $\bar{\alpha}_I$ variables is far too large to formulate (P) explicitly even for moderate m. A classic way to solve programs with large numbers of variables is to apply delayed column generation (Bertsekas 1999). In fact, a now standard way to design iterative combinatorial auctions is to formulate the selection problem with variables for each allocation as in (P), and apply a primal-dual or subgradient algorithm with delayed column generation to reach a solution (Bikhchandani et al. 2001). This approach has proven effective in practice (Parkes and Ungar 2000). The same issue arises with support vector machines over large datasets. There, cutting plane algorithms have been developed to perform tasks such as support vector regression (Joachims 2006). Given the strong resemblance between our problem and support vector machines, it should be possible to leverage ideas from that literature as well. This is the subject of ongoing work.

Incentive Compatibility

We show in this section that our auction is approximately incentive compatible when it charges payments r. Let κ be a constant such that $\|\phi(x)\| \leq \kappa$ for all $x \in X$. This constant can be seen as a crude measure of the "complexity" of the associated price structure: *smaller* κ will correspond to *more complex* price structures. For instance, we have $\kappa = \sqrt{m}$ for the linear kernel and $\kappa = 1$ for the identity kernel. Some proofs in this section appeal to results from convex analysis that can be found in (Bertsekas 1999), abbreviated (B) in the proofs.

Stability We begin by showing that for large enough λ, the presence or absence of a buyer h does not impact the optimal dual solution (the price vector) significantly. We first formulate the dual problem in a slightly more convenient form. The *surplus functions* of buyer i and the seller are defined as follows:

$$\pi_i(p) = \max\{v_i(x_i) - p(x_i), 0\} \quad (3)$$

$$\bar{\pi}(p) = \max_{I \in \mathcal{F}} \sum_{i \in I} p(x_i) \quad (4)$$

These functions are convex, being the maximum of linear functions of p. We have the following bound on their subgradients.

Lemma 1 *For a single-minded buyer i, we have $\|\pi_i'\| \leq \|\phi(x_i)\|$ for any p and $\pi_i' \in \partial\pi_i(p)$.*

Proof. Let $f(p_i) = v_i(x_i) - p_i$ for $p_i \in \mathbf{R}$ and let $g(p) = f(\langle\phi(x_i), p\rangle)$ for $p \in \mathbf{R}^M$. Note that $\pi_i(p) = \max\{g(p), 0\}$. We have $\partial f(p_i) = -1$ for all p_i, so by (B, B.24(e)) $\partial g(p) = -\phi(x_i)$. Thus by (B, B.25(b)), $\partial\pi_i(p)$ is either $\{0\}$, $\{-\phi(x_i)\}$, or the convex combination of these two. No matter what the case the norm of a subgradient is always at most $\|\phi(x_i)\|$. □

The dual problem can now be formulated as that of maximizing the function

$$V(p) = \sum_i \pi_i(p) + \bar{\pi}(p) + \frac{\lambda}{2}\|p\|^2.$$

This function is convex, being the sum of convex functions. Let $V^{-h}(p)$ denote the same function but with buyer h removed (i.e., the $\pi_h(p)$ term is subtracted away).

Proposition 1 *If p is an optimal dual solution with all buyers present, and p^{-h} is an optimal dual solution with buyer h removed, then*

$$\left\|p - p^{-h}\right\| \leq \frac{\kappa}{\lambda}.$$

Proof. By (B, B.24(f)), we have $\mathbf{0} \in \partial V(p)$ and $\mathbf{0} \in \partial V^{-h}(p^{-h})$. By (B, B.24(b)) and (B, B.24(d)) therefore, there are subgradients $\pi_i'(p) \in \partial\pi_i(p)$, $\bar{\pi}'(p) \in \partial\bar{\pi}(p)$, $\pi_i'(p^{-h}) \in \partial\pi_i(p^{-h})$, and $\bar{\pi}'(p^{-h}) \in \partial\bar{\pi}(p^{-h})$ such that

$$\sum_i \pi_i'(p) + \bar{\pi}'(p) + \lambda p = \mathbf{0} \quad (5)$$

$$\sum_{i \neq h} \pi_i'(p^{-h}) + \bar{\pi}'(p^{-h}) + \lambda p^{-h} = \mathbf{0} \quad (6)$$

Subtracting (6) from (5), then taking the inner product with $p - p^{-h}$ and rearranging, we have

$$\begin{aligned} 0 &= \sum_{i \neq h} \langle \pi_i'(p) - \pi_i'(p^{-h}), p - p^{-h} \rangle \\ &+ \langle \bar{\pi}'(p) - \bar{\pi}'(p^{-h}), p - p^{-h} \rangle \\ &+ \langle \pi_h'(p), p - p^{-h} \rangle + \lambda \left\| p - p^{-h} \right\|^2 . \end{aligned}$$

By the monotonicity property of subgradients (an easy consequence of their definition) we have $\langle \pi_i'(p) - \pi_i'(p^{-h}), p - p^{-h} \rangle \geq 0$ for each buyer $i \neq h$ and the same for the seller term. Thus we have $\langle \pi_h'(p), p - p^{-h} \rangle + \lambda \left\| p - p^{-h} \right\|^2 \leq 0$. Continuing,

$$\begin{aligned} \lambda \left\| p - p^{-h} \right\|^2 &\leq \langle \pi_h'(p), p^{-h} - p \rangle \\ &\leq \| \pi_h'(p) \| \left\| p - p^{-h} \right\| \\ &\leq \| \phi(x_h) \| \left\| p - p^{-h} \right\| \\ &\leq \kappa \left\| p - p^{-h} \right\| . \end{aligned}$$

The third inequality follows from Lemma 1, and the second follows from the Cauchy-Schwarz inequality. This completes the proof. □

It may seem curious that Proposition 1 does not depend on $v_h(x_h)$. After all, if buyer h has a high value, then its absence should have a higher impact on the objective and thus on the solution than the absence of low value buyers. This reasoning is misleading for the following reason. Consider the dual objective with a very large λ relative to the values. In this case the quadratic program will essentially try to minimize $\|p\|$, no matter what the values. Thus the proximity of p and p^{-h} is expressed in terms of λ alone. We see from the proof though that the size of bundle x_h does impact the proximity of the two price vectors, and this is the origin of the κ term.

Universal Competitiveness The connection between stability and approximate incentive compatibility comes via the concept of universal competitive equilibrium, introduced by Mishra and Parkes (2007). Prices p are *universally competitive* if they support not only the efficient allocation, but also any efficient allocation y^{-h} that arises when buyer h is removed, for any buyer h. We show that our program computes approximately universally competitive prices. We believe this result is interesting in its own right.

Proposition 2 *Let p be the supporting prices computed by the dual. Let y^{-h} be the efficient allocation computed when buyer h is removed, and let y' be an arbitrary allocation among the agents with h removed. Then for each buyer $i \neq h$ and for the seller, we have*

$$\begin{aligned} v_i(y_i^{-h}) - p(y_i^{-h}) + \frac{\kappa^2}{\lambda} &\geq v_i(y_i') - p(y_i') \\ \sum_{i \neq h} p(y_i^{-h}) + \frac{(n-1)\kappa^2}{\lambda} &\geq \sum_{i \neq h} p(y_i') \end{aligned}$$

Proof. We have the following derivation.

$$\begin{aligned} & v_i(y_i') - p(y_i') \\ = \ & v_i(y_i') - p^{-h}(y_i') + p^{-h}(y_i') - p(y_i') \\ \leq \ & v_i(y_i^{-h}) - p^{-h}(y_i^{-h}) + p^{-h}(y_i') - p(y_i') \\ = \ & v_i(y_i^{-h}) - p(y_i^{-h}) + p(y_i^{-h}) - p^{-h}(y_i^{-h}) \\ & + p^{-h}(y_i') - p(y_i') \\ = \ & v_i(y_i^{-h}) - p(y_i^{-h}) + \langle p^{-h} - p, \phi(y_i') - \phi(y_i^{-h}) \rangle \\ \leq \ & v_i(y_i^{-h}) - p(y_i^{-h}) + \left\| p^{-h} - p \right\| \left\| \phi(y_i') - \phi(y_i^{-h}) \right\| \\ \leq \ & v_i(y_i^{-h}) - p(y_i^{-h}) + \frac{\kappa^2}{\lambda}. \end{aligned}$$

The first inequality follows from the fact that prices p^{-h} support the allocation $y^{-h}, \bar{y}^{-h}$. The second follows from the Cauchy-Schwarz inequality, and the last from Proposition 1. The line of argument is completely analogous for the seller's revenue bound. □

Mishra and Parkes (2007) showed that universally competitive prices, besides being competitive, also contain all the information necessary to compute VCG payments. From the approximately universally competitive prices p computed by our auction, we will therefore be able to derive approximate VCG payments and achieve approximate incentive compatibility.

VCG Payments We are now ready to prove our main result, characterizing how well payments r approximate VCG payments.

Proposition 3 *For each buyer i, the payments q and r satisfy*

$$r_i \geq q_i \geq r_i - \frac{2(n-1)\kappa^2}{\lambda}.$$

Proof. Let y be the efficient allocation computed by the auction together with prices p, let y^{-h} be the efficient allocation computed when buyer h is removed, and let $\tilde{y}_i^{-h}$ be a revenue-maximizing allocation among buyers $i \neq h$ with respect to p. Note that for any buyer $i \neq s$,

$$\begin{aligned} v_i(y_i^{-h}) - v_i(y_i) &= [v_i(y_i^{-h}) - p(y_i^{-h})] - [v_i(y_i) - p(y_i)] \\ &\quad + p(y_i^{-h}) - p(y_i) \\ &\geq p(y_i^{-h}) - p(y_i) - \frac{\kappa^2}{\lambda}. \qquad (7) \\ &\leq p(y_i^{-h}) - p(y_i). \qquad (8) \end{aligned}$$

where (7) follows from Proposition 2 and (8) by the fact that p supports y. Summing (8) over all $i \neq h$ combined with the fact that $\tilde{y}_i^{-h}$ maximizes revenue at prices p (among $i \neq h$) yields the upper bound. Summing (7) over all $i \neq h$ combined with the seller bound in Proposition 2 yields the lower bound. □

Approximate incentive compatibility now follows almost immediately.

Theorem 1 *The auction that charges payments r is ϵ-incentive compatible for $\epsilon = \frac{2(n-1)\kappa^2}{\lambda}$.*

Proof. Let (y, r) be the allocation and payments computed when buyer h reports its value function truthfully, and let (y', r') be the resulting allocation and payments when it reports some other value. We have:

$$\begin{aligned} v_h(y_h) - r_h &\geq v_h(y_h) - q_h - \frac{2(n-1)\kappa^2}{\lambda} \\ &\geq v_h(y'_h) - q'_h - \frac{2(n-1)\kappa^2}{\lambda} \\ &\geq v_h(y'_h) - r'_h - \frac{2(n-1)\kappa^2}{\lambda}. \end{aligned}$$

The first inequality follow from the lower bound in Proposition 3, the second from the incentive compatibility of the VCG payment scheme, and the third from the upper bound in Proposition 3. □

Discussion

The error ϵ in Theorem 1 can be reduced in two ways: by using a price structure with high complexity (small κ) or a large penalty term (large λ). But note that both cannot be improved in tandem. With more complex price structures, the allocation problem becomes increasingly difficult and one needs an increasingly small λ to achieve feasibility within a reasonable tolerance (Lahaie 2009).

Holding k and hence κ fixed, one way to maximize λ is to begin by setting it large, and then solve (P) over rounds using updates $\lambda \leftarrow \tau\lambda$, where $\tau \in [0, 1)$, until the requisite tolerance on feasibility is achieved. This is precisely the approach taken by penalty methods for solving programs such as ours (Bertsekas 1999). Taking τ close to 1 will improve the minimization, but also increase the computation time.

To minimize the error in Theorem 1, one could then take the following approach. Consider price structures of increasing complexity, each time minimizing λ as just described to obtain the best possible error ϵ. There are common kernels, such as the Gaussian and polynomial kernels (Schölkopf and Smola 2001), that have parameters allowing one to adjust their complexity. Eventually one will reach a point of diminishing returns, settling on the optimal trade-off between the complexity and penalty terms. This approach is analogous to that of structural risk minimization in statistical learning theory (Vapnik 1998), where one trades off the ability of a kernel to fit training data with its ability to generalize.

Finally, let us examine how closely the price $p(y_h)$ buyer h faces can match its approximate VCG payment r_h. Note that $p_h - r_h = \sum_i p(y_i) - \sum_{i \neq h} p(y_i^{-h}) \geq 0$, so an agent's payment never exceeds its price (recall that p supports y). By the Cauchy-Schwarz inequality, this difference is bounded by $\|p\|$ multiplied by the term $\delta = \left\| \sum_i \phi(y_i) - \sum_{i \neq h} \phi(y_i^{-h}) \right\|$. The latter quantity can be seen as an indicator of the distance between prices and payments. For instance, note that with the linear kernel, we have $\sum_i \phi(y_i) = \sum_{i \neq h} \phi(y_i^{-h}) = \mathbf{1}$, so $\delta = 0$. This means that if our auction can successfully find an integer solution with this kernel, then the resulting prices satisfy $p(y_h) = r_h$ and correspond to approximate VCG payments. On the other hand, we see that δ can be as large as $\sqrt{2n-1}$ with the identity kernel. This suggests that prices and payments diverge with increasingly complex price structures. In general, the development of new complexity measures to characterize economic properties of kernels appears to be a stimulating avenue for future research.

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